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Singularity avoidance in quantum FRW cosmologies in the presence of barotropic perfect fluids

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Abstract
Recently, an effective formulation of gravity which lies in between the Wheeler–DeWitt approach and classical cosmology was discussed. It was shown that the Big Bang singularity of FRW cosmologies is avoided in a quite natural way. Here, we aim to prove that this formulation is able to avoid the Big Rip singularity too, in contradistinction with Schutz's formalism as applied to quantum cosmological perfect fluids. Actually, in using this last formalism, some authors have argued that such singularity would persist even after quantization, however, what we carried out, with our formulation as a guide, proved not to be the case. Also, it will be argued that it is the implicit regularization of the classical Hamiltonian performed in loop quantum cosmology, which is needed in loop cosmology in order to build a well-defined quantum (discrete) theory, which avoids the Big Rip singularity in that theory, this mechanism being different from other, ordinarily invoked quantum effects.

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1. Introduction

In a recent paper [1], we have developed an effective theory of quantum gravity that avoids the Big Bang singularity. This modified gravity quantization stems from the Wheeler–DeWitt equation: $\hat{H}\Phi = 0$ [2], with $\hat{H}$ the quantum Hamiltonian. To address the question of a possible singularity at finite time, we considered an effective formulation given in terms of the following Schrödinger equation (where, as for time, the cosmic one was chosen), with additional conditions, namely

$$\mathrm{i}\hbar\partial_t\Phi(t) = \hat{H}\Phi(t), \qquad \Phi(t^*) = \Psi, \qquad \langle\hat{H}\rangle_\Psi = 0, \ \|\Psi\| = 1. \tag{1}$$

We have adopted here the same point of view, about the problem of time in quantum cosmology, as Carroll's in [3]. More precisely, it has often been argued that time does not exist. Here we take the contrary perspective, we imagine that time exists and that our universe is described by a quantum state which evolves following the rules of ordinary time-dependent quantum mechanics.

It is important to stress that our study concentrates on the scale factor as the only dynamical factor in a quantum setting. Also, we do not take into consideration that the theory of General Relativity is very likely altered when gravity becomes strong at singularities (and string theory, supersymmetry considerations, etc should be taken into account). In this sense, our proposal may be considered a toy model designed to illustrate, in a controlled and clear way, some specific issues concerning the important problem of the singularities. Our approach and the results obtained will then be compared with other existing alternatives.

It is well known that the quantum Hamiltonian $\hat{H}$, obtained with the usual rules of quantum mechanics, is generically symmetric but not self-adjoint. Von Neumann's theorem [4, 5] allows it to be extended (sometimes in infinitely many ways) to a self-adjoint operator. Stone's theorem then applies, leading to a solution which is valid for all times t and, consequently, one can then compute the average of the quantum operator $\hat{a}$ corresponding to the classical scale factor a. That is, one calculates the following effective scale factor $a_{\text{eff}}(t) \equiv \langle\Phi(t)|\hat{a}\Phi(t)\rangle$, where $\Phi(t)$ is the solution of the effective Schrödinger equation above. It is not difficult to see that if $\Phi(t)$ belongs to the domain of the operator $\hat{a}$ at any time, then the effective scale factor $\langle\Phi(t)|\hat{a}\Phi(t)\rangle$ is always strictly positive, and one can conclude that the Big Bang singularity is avoided. Physically, the self-adjoint extension of the Hamiltonian operator corresponding to a FRW cosmology can be visualized assuming that there is an infinite barrier potential at the point $a = 0$ and then, when the effective factor scale approaches zero, at some finite time it suddenly bounces back and starts increasing.

In the present paper, we will apply this effective formulation to the case of a barotropic fluid where we can clearly see what the physical meaning of the self-adjoint extension of the symmetric operator actually is. By performing a canonical transformation and by applying Weyl's limit point-limit circle criterion, we will show how our effective formulation is able to avoid both the classical Big Bang and the Big Rip singularities.

Finally, we will compare our approach with Schutz's formalism as applied to quantum cosmological fluids [6–8] and will show the equivalence of both approaches when one uses the same time variable—which depends on the fluid's equation of state. In order to later express the results in cosmic time, one has to perform an appropriate change of time variable. However, if it is not carried out accurately, this transformation easily gives rise to some paradoxical results, as the non-avoidance of the Big Rip singularity after quantization, which, in our opinion, is not understandable. What this means is that such method should not be the correct way to quantize cosmological fluids in cosmic time; rather, the right procedure is to employ the *effective* formulation in cosmic time. We also establish a comparison with loop quantum cosmology (LQC) eventually showing that in LQC it is the regularization of the classical Hamiltonian, performed invoking the quantum (discrete) nature of the geometry [9–11], what is able to avoid the Big Rip singularity, rather than quantum effects themselves.

To conclude this introduction, we want to stress that we only study the avoidance of singularities in quantum FRW models (minisuperspace models), that is, for homogeneous and isotropic geometries. This study is important because one can obtain analytic results, and it may help us to understand the problem of singularities in general relativity. However, this last issue is far from being solved, because we do not have an underlying quantum theory of general relativity. For that reason, solvable models, like FRW models of the kind considered here, can be useful to understand the avoidance of singularities in the more general case.

2. Effective formulation for a barotropic perfect fluid

In this section, we apply our effective formulation to the case of a barotropic perfect fluid with equation of state $p = \omega\rho$. The Lagrangian of the system, in terms of the cosmic time, is

$$L = \frac{\gamma^2}{2}(c^2 k - \dot{a}^2)a - \rho(a)a^3, \tag{2}$$

where k is the three-dimensional curvature and $\gamma^2 \equiv \frac{3c^2}{4\pi G} = \frac{3\hbar}{4\pi c l_p^2}$ being G Newton's constant and l_p the Planck length.

The momentum and Hamiltonian are, respectively, $p_a = -\gamma^2\dot{a}a$ and $H = -\frac{1}{2\gamma^2 a}p_a^2 - \frac{\gamma^2 c^2}{2}ka + \rho(a)a^3$. Using the conservation equation $\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p)$, we have $\rho(a) = \rho_0(a/a_0)^{-3(\omega+1)}$. Then, the dynamical equations become

$$\dot{a} = -\frac{p_a}{\gamma^2 a}; \qquad \dot{p}_a = -\frac{p_a^2}{2\gamma^2 a^2} + \frac{\gamma^2 c^2}{2}k + 3\omega\rho(a)a^2, \tag{3}$$

with the constraint $H = 0$. The quantization rule

$$g^{AB}p_A p_B \longrightarrow -\hbar^2\nabla_A\nabla^A = -\frac{\hbar^2}{\sqrt{|g|}}\partial_A(\sqrt{|g|}g^{AB}\partial_B) \tag{4}$$

yields the following Hamiltonian operator

$$\hat{H} = \frac{\hbar^2}{2\gamma^2 a}\partial_a^2 - \frac{\gamma^2 c^2}{2}ka + \rho(a)a^3, \tag{5}$$

which is symmetric with respect to the usual inner product, $\langle\Phi|\Psi\rangle = \int_0^\infty daa\Phi^*(a)\Psi(a)$, of the Hilbert space $\mathcal{L}^2((0,\infty), ada)$.

In order to simplify the quantization, it is advisable to perform the canonical transformation [12, 13]:

$$x \equiv \frac{2\gamma}{3}a^{3/2}, \qquad p \equiv \frac{1}{\gamma}a^{-1/2}p_a, \tag{6}$$

and then the Hamiltonian is given by

$$H(x,p) = -\frac{p^2}{2} - \frac{\gamma^2 c^2}{2}k\left(\frac{3x}{2\gamma}\right)^{2/3} + \frac{9}{4\gamma^2}\widetilde{\rho}(x)x^2, \tag{7}$$

where now $\widetilde{\rho}(x) = \rho_0(x/x_0)^{-2(\omega+1)}$. In these variables the Hamiltonian acquires the simple form

$$\hat{H} = \frac{\hbar^2}{2}\partial_x^2 + V(x), \tag{8}$$

with $V(x) \equiv -\frac{\gamma^2 c^2}{2}k\left(\frac{3x}{2\gamma}\right)^{2/3} + \frac{9}{4\gamma^2}\widetilde{\rho}(x)x^2$. Again, the inner product is the ordinary one defined in the space $\mathcal{L}^2((0,\infty), \mathrm{d}x)$. Applying Weyl's limit point-limit circle criterion and the Frobenius method for second-order differential equations (details are given in [14]), we can deduce that, when $-1 \leqslant \omega \leqslant 1$, both deficiency indices are $(1, 1)$; this means that the different self-adjoint extensions are parameterized by the following boundary condition: $\Psi(0) = r\Psi'(0)$ [15]. When $\omega < -1$, both deficiency indices are $(2, 2)$ and the different self-adjoint extensions are parameterized by a unitary 2×2 matrix, what means that the self-adjoint extensions are defined by boundary conditions both at 0 and also at $+\infty$. As a consequence, the eigenfunctions belong in the Hilbert space and the spectrum is purely discrete [15]. For the case $w > 1$, we can only affirm that there exists at most one self-adjoint extension.

Once we have extended our symmetric operator $\frac{\hbar^2}{2}\partial_x^2 + V(x)$ to a self-adjoint one, namely $\hat{H}_{\mathrm{SA}}$, our effective formulation can be written as follows (t being the cosmic time):

$$\mathrm{i}\hbar\partial_t\Phi(t) = \frac{\hbar^2}{2}\partial_x^2\Phi(t) + V(x)\Phi(t), \tag{9}$$

with the additional conditions $\Phi(t_0) = \Psi$, $\langle\hat{H}_{\mathrm{SA}}\rangle_\Psi = 0$, $\|\Psi\| = 1$, and the consistency condition $a_{\mathrm{eff}}(t_0) = a_0$. With this effective formulation, we will construct an analytic solution. This is only possible to do when one considers a dust fluid ($\omega = 0$) in the flat case $k = 0$. We use the notation

$$C \equiv \frac{9}{4\gamma^2}\rho_0 x_0^2, \qquad K \equiv \frac{\hbar^2}{2}, \tag{10}$$

and we assume that our self-adjoint extension is defined imposing that the wavefunction vanishes at the origin. Then an analytic solution of the above equation is

$$\Phi(x,t) = Bx\left(\frac{1}{\delta} - \mathrm{i}\frac{t}{\hbar}\right)^{-3/2} \mathrm{e}^{-\frac{x^2}{4K\left(\frac{1}{\delta}-\mathrm{i}\frac{t}{\hbar}\right)}}\,\mathrm{e}^{-\mathrm{i}\frac{Ct}{\hbar}}, \tag{11}$$

with B and δ real constants. We now impose the normalization of the wavefunction, $\|\Phi(0)\| = 1$, and obtain the relation

$$B^2\sqrt{\frac{\delta^3K^3\pi}{2}} = 1, \tag{12}$$

and, finally, we need to impose $\langle\hat{H}\rangle_{\Phi(0)} = 0$ to obtain

$$B^2\delta^3\frac{4}{11}\sqrt{\frac{K\pi}{2\delta}} = C/K. \tag{13}$$

Solving these two equations, one has $\delta = \frac{4C}{11}$ and $B = \left(\frac{2.11^3}{4^3C^3K^3\pi}\right)^{1/4}$. In this way, we have obtained a solution of our effective formulation given by (11). With this solution it is now easy to calculate

$$a_{\mathrm{eff}}(t) = \left(\frac{3}{2\gamma}\right)^{2/3}\langle x^{2/3}\rangle_{\Phi(t)} = \sqrt{2/\pi}\,2^{5/6}(K\delta)^{1/3}\left(\frac{3}{2\gamma}\right)^{2/3}\left(\frac{1}{\delta^2} + \frac{t^2}{\hbar^2}\right)^{1/3}\Gamma(11/6), \tag{14}$$

which shows that, for large values of $|t|$, the behavior is $a_{\mathrm{eff}}(t) \propto t^{2/3}$, as the classical one. Note that this also proves that the universe bounces at $t = 0$, with a scale factor $a_{\mathrm{eff}}(0) \propto l_p\left(\frac{m_pc^2}{\rho_0a_0^3}\right)^{1/3}$, m_p being Planck's mass. To be consistent, we have to identify $a_{\mathrm{eff}}(0)$ with a_0, this means that a_0 is not a free parameter, it has the following value $a_0 \propto \sqrt{l_p}\left(\frac{m_pc^2}{\rho_0}\right)^{1/6}$, and if one chooses ρ_0 of the same order that the Planck density one has $a_0 \propto l_p$.

Another analytic solution can be built, if one considers the extension defined by the boundary condition $\Psi'(0) = 0$. This solution is (see for details [12])

$$\Phi(x,t) = (8b/\pi)^{1/4}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\rho_0a_0^3t}(1 + 2\mathrm{i}\hbar\beta t)^{-1/2}\,\mathrm{e}^{-\frac{\beta x^2}{1+2\mathrm{i}\hbar\beta t}}, \qquad \beta \equiv b + \mathrm{i}B. \tag{15}$$

This function is normalized; now imposing $\langle\hat{H}\rangle_{\Phi(0)} = 0$, one obtains

$$\frac{\hbar^2(b^2 + B^2)}{b} = \rho_0a_0^3. \tag{16}$$

The average of the operator $\hat{x}$ is given by

$$\langle \hat{x} \rangle_{\Phi(t)} = (2\pi b)^{-1/2}[1 - 4B\hbar t + 4(b^2 + B^2)\hbar^2 t^2]^{1/2}, \tag{17}$$

what shows that the universe bounces at $t = \frac{B}{2\hbar(b^2+B^2)}$.

To be consistent, we have to impose $\langle \hat{x} \rangle_{\Phi(0)} = x_0$, and one has $b = \frac{1}{2\pi x_0^2} = \frac{9}{8\pi\gamma^2 a_0^3}$. Then for large values of a_0 one has $B \cong -\frac{3}{2\gamma\hbar}\sqrt{\frac{\rho_0}{2\pi}}$ (we have chosen the negative sign because we have assumed that the universe expands at $t = 0$) and, consequently,

$$\langle \hat{x} \rangle_{\Phi(t)} \cong x_0 \left[1 + \frac{6}{\gamma}\sqrt{\frac{\rho_0}{2\pi}}t + \frac{9\rho_0}{2\pi\gamma^2}t^2\right]^{1/2}. \tag{18}$$

It should be noted that in the quantum theory we have associated with the classical scale factor a an operator denoted by $\hat{a}$ that we call 'scale factor operator', in the same way that one introduces the position operator in no-relativistic quantum mechanics. In quantum mechanics, the position has no physical meaning; however, when fluctuations can be neglected, e.g. in the classical limit, one can identify the average of this operator with the classical position of a particle. This is what happens in our model, when the system is far from the singularity and the fluctuations can be neglected, one can identifiy the effective scale factor with the classical one and, then, the effective scale factor satisfies *all* the classical properties. For example, in the case $k = 0$, during the classical regime, the scaling $a_{\mathrm{eff}}(t_0) \to Ca_{\mathrm{eff}}(t_0)$, where C is some constant, implies $a_{\mathrm{eff}}(t) \to Ca_{\mathrm{eff}}(t)$.

To conclude this section, we show how our effective formulation actually solves, both the classical Big Bang and the Big Rip singularities. The first appears when $w > -1$, in which case the classical solution is $a(t) = a_0(t/t_0)^{\frac{2}{3(1+\omega)}}$, what shows that the classical solution is only defined in the interval $(0, \infty)$ (a classical Big Bang singularity). However, Stone's theorem assures that the solution of our effective problem exists for any time value and, also, that the effective factor scale (the average of the scale factor operator) will never vanish.

The classical Big Rip singularity appears, on the other hand, when $\omega < -1$. The classical solution, in the flat case, reads $a(t) = a_0\left(\frac{t-t_s}{t_0-t_s}\right)^{\frac{2}{3(1+\omega)}}$, with t_s finite. In this situation, the singularity occurs at time t_s, because the factor scale diverges, and then the classical solution is only defined in the domain $(-\infty, t_s)$. As above, Stone's theorem states that the quantum solution of our effective formulation exists for all time. Moreover, to ensure that the effective scale factor does not diverge at finite time, we first note that, in the flat case, when $\omega < -1$, the spectrum is discrete. Then, we may consider the following superposition: $\Phi(x, t) = \sum_j \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\lambda_j t}\alpha_j \Phi_j(x)$, where λ_j are the eigenvalues and $\Phi_j(x)$ are the respective, normalized eigenfunctions. The normalization of the wave packet yields the condition $\sum_j |\alpha_j|^2 = 1$. The effective scalar factor is $a_{\mathrm{eff}}(t) \equiv \left(\frac{3}{2\gamma}\right)^{2/3} \int_0^\infty \mathrm{d}x |\Phi(x, t)|^2 x^{2/3} = \left(\frac{3}{2\gamma}\right)^{2/3} \sum_{j,k} \alpha_j \alpha_k \,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(\lambda_j - \lambda_k)t} \int_0^\infty \mathrm{d}x\, \Phi_j(x)\Phi_k^*(x)x^{2/3}$. Then, taking into account, in the flat case, that the behavior of the eigenfunctions at ∞ is $|\Phi_j(x)| \sim x^{\omega/2}$ (see, for details, [15] p. 75), we obtain, for $\omega < -5/3$, the following bound:

$$\int_0^\infty \mathrm{d}x\, \Phi_j(x)\Phi_k^*(x)x^{2/3} = \int_0^{\bar{x}} \mathrm{d}x\, \Phi_j(x)\Phi_k^*(x)x^{2/3} + \int_{\bar{x}}^\infty \mathrm{d}x\, \Phi_j(x)\Phi_k^*(x)x^{2/3}$$
$$\leqslant \bar{x}^{2/3} + \int_{\bar{x}}^\infty \mathrm{d}x\, x^\omega x^{2/3} = \bar{x}^{2/3} + (-\omega - 5/3)^{-1}\bar{x}^{5/3+\omega}, \tag{19}$$

for some conveniently large value $\bar{x}$. With this bound—always in the flat case—it is easy to show that, for $\omega < -5/3$, one has $a_{\mathrm{eff}}(t) \leqslant 2(\bar{x}^{2/3} + (-\omega - 5/3)^{-1}\bar{x}^{5/3+\omega})$, for any t. Now, when $-5/3 \leqslant \omega < -1$ one actually needs to know the explicit form of the wave packet; but it seems clear that, in this last situation, the effective scalar factor will be bounded too.

3. Comparison with others approaches

In this section, we will compare our approach with Schutz's formalism as applied to quantum cosmological fluids. The Hamiltonian in Schutz's formalism is (we use units $\hbar = c = 16\pi G = 1$) [6–8]

$$H = -\frac{1}{24a}p_a^2 - 6ka + \frac{p_T}{a^{3\omega}}, \tag{20}$$

where p_T is the canonical momentum associated with matter, and the connection between the cosmic time t and T is given by $\mathrm{d}t = a^{3\omega}\,\mathrm{d}T$. Comparison of (10) with our Hamiltonian yields $p_T = \rho_0 a_0^{3(\omega+1)}$. In this formalism, the Wheeler–DeWitt equation reads ($p_T \to -i\partial_T$)

$$\frac{1}{24a}\partial^2_{a^2}\Phi - 6ka\Phi = \frac{\mathrm{i}}{a^{3\omega}}\partial_T\Phi \longleftrightarrow \mathrm{i}\partial_T\Phi = \frac{a^{3\omega-1}}{24}\partial^2_{a^2}\Phi - 6ka^{3\omega+1}\Phi. \tag{21}$$

To obtain the effective formulation in terms of time T, one has to use the Lagrangian

$$L = 6a^{3\omega}\left(k - \frac{a'^2}{a^{6\omega}}\right)a - \rho(a)a^{3(\omega+1)}, \tag{22}$$

where $a' \equiv \frac{\mathrm{d}a}{\mathrm{d}T}$. Then, the Hamiltonian is

$$H = -\frac{a^{3\omega-1}}{24}p_a^2 - 6ka^{3\omega+1} + \rho(a)a^{3(\omega+1)}, \tag{23}$$

and the effective formulation reads now

$$\mathrm{i}\partial_T\Phi = \frac{a^{3\omega-1}}{24}\partial^2_{a^2}\Phi - 6ka^{3\omega+1}\Phi + \rho(a)a^{3(\omega+1)}, \tag{24}$$

with the additional conditions $\Phi(T^*) = \Psi$, $\langle\hat{H}_{\mathrm{SA}}\rangle_\Psi = 0$, $\|\Psi\| = 1$. Note that the term $\rho(a)a^{3(\omega+1)}$ is constant; thus, we can perform the change $\Phi = \widetilde{\Phi}\,\mathrm{e}^{-\mathrm{i}\rho_0 a_0^{3(\omega+1)}T}$ in (24) to obtain equation (21), what rigorously shows the equivalence of both quantization methods.

We now study the self-adjoint extensions of the Hamiltonian operator that appears in equation (21), in the flat case. Once more, using Weyl's limit circle-limit point criterion and the Frobenious method, it is not difficult to show that, for $\omega \leqslant 1$, the deficiency indices are $(1, 1)$. This means that, in the first case, the self-adjoint extensions are parameterized by a boundary condition at 0. One can calculate the eigenfunctions of the Hamiltonian explicitly. Writing $\Phi(a, T) = \mathrm{e}^{\mathrm{i}ET}\phi_E(a)$ and choosing for boundary condition at zero, $\Phi(0) = 0$, we get (for $\omega < 1$)

$$\Phi_E = \sqrt{a}J_{\frac{1}{3(1-\omega)}}\left(\frac{\sqrt{96E}}{3(1-\omega)}a^{3(1-\omega)/2}\right), \qquad \text{for} \quad E > 0, \tag{25}$$

where J denotes the Bessel function of the first kind. An analytic solution is easily found using the eigenfunctions given in (25) [7, 8]:

$$\Phi(a, T) = a\frac{\mathrm{e}^{-\frac{a^{3(1-\omega)}}{4B}}}{(-2B)^{\frac{4-3\omega}{3(1-\omega)}}}, \tag{26}$$

with $B \equiv \delta - \frac{3}{32}\mathrm{i}(1-\omega)^2T$, where δ is a free real parameter which, in our formulation, has to be chosen in order to satisfy the condition $\langle H\rangle_\Phi = 0$. Using this solution, the effective scalar factor turns out to be

$$a_{\mathrm{eff}}(T) = \frac{\Gamma\left(\frac{5-3\omega}{3(1-\omega)}\right)}{\Gamma\left(\frac{4-3\omega}{3(1-\omega)}\right)}(2\delta)^{\frac{1}{3(1-\omega)}}\left[\frac{9(1-\omega)^4}{(32\delta)^2}T^2 + 1\right]^{\frac{1}{3(1-\omega)}}, \qquad \forall T \in \mathbb{R}. \tag{27}$$

In Schutz's formalism, to obtain the wave packet as a function of the cosmic time, some authors perform the time change $\mathrm{d}t = a_{\text{eff}}^{3\omega}(T)\,\mathrm{d}T$ [8, 16]. Thus, the wave packet is given by $\Phi(a, T(t))$, with the Schrödinger equation in the cosmic time being

$$\mathrm{i}\partial_t \Phi = \frac{1}{a_{\text{eff}}^{3\omega}(T(t))}\left(\frac{a^{3\omega-1}}{24}\partial_{a^2}^2\Phi - 6ka^{3\omega+1}\Phi\right). \tag{28}$$

We shall now show that this change of variable give rise to paradoxical results. For instance, in the flat case, when one considers a phantom fluid ($\omega < -1$), one has $t_{\text{BR}} \equiv \int_0^\infty a_{\text{eff}}^{3\omega}(s)\,\mathrm{d}s < +\infty$ [16]. This means that the Schrödinger equation is only defined for the cosmic time in the interval $[-t_{\text{BR}}, t_{\text{BR}}]$ and, also, that the effective scalar factor diverges at cosmic time $t = t_{\text{BR}}$, that is, the Big Rip singularity survives after quantization. However, we have already seen that the Big Rip singularity is actually avoided in the effective formulation, what means, from our viewpoint, that the change of variable $\mathrm{d}t = a_{\text{eff}}^{3\omega}(T)\,\mathrm{d}T$ may not be the correct one to perform in order to obtain the Schrödinger equation in cosmic time. The proposed right procedure would be, namely, to start with the Lagrangian (2), to obtain then the corresponding classical Hamiltonian and, finally, to apply the standard quantization rules with the quantum constraint $\langle \hat{H}_{\text{SA}}\rangle_\Phi = 0$.

3.1. Big Rip singularity avoidance in loop quantum cosmology

We will now show how loop quantum cosmology is able to avoid the Big Rip singularity. Consider the canonically conjugate variables $V \equiv a^3$ and $\beta \equiv \iota\dot{a}/a$, which satisfy $\{\beta, V\} = \frac{3\iota}{\gamma^2}$, where ι denotes the Barbero-Immirzi parameter [17, 18]. Consider also the holonomies $h_j(\lambda) \equiv \mathrm{e}^{-\mathrm{i}\frac{\lambda\beta}{2c}\sigma_j}$, where σ_j are the Pauli matrices and λ is a parameter with dimensions of length. The general formulas of loop quantum gravity (LQG) can be used to obtain the following gravitational, regularized Hamiltonian (which is needed in order to construct a well-defined quantum theory) [11, 19, 20]:

$$\begin{aligned} H_{\text{grav,reg}} &\equiv -\frac{\hbar^2 c}{32\pi^2 l_p^4 \iota^3}\frac{V}{\lambda^3}\sum_{i,j,k}\varepsilon^{ijk}Tr[h_i(\lambda)h_j(\lambda)h_i^{-1}(\lambda)h_j^{-1}(\lambda)h_k(\lambda)\{h_k^{-1}(\lambda), V\}] \\ &= -\frac{\gamma^2 c^2}{2\iota^2\lambda^2}V\sin^2\frac{\lambda\beta}{c}. \end{aligned} \tag{29}$$

Taking this regularized Hamiltonian as the gravitational part of the full Hamiltonian, this last one will be given by [17]

$$H_{\text{reg}} \equiv -\frac{\gamma^2 c^2}{2\iota^2\lambda^2}V\sin^2\frac{\lambda\beta}{c} + a^3\rho, \tag{30}$$

and the dynamical equation for the scalar factor reads

$$\dot{a} = \{a, H_{\text{reg}}\} = \frac{ca}{2\lambda\iota}\sin\frac{2\lambda\beta}{c}. \tag{31}$$

Imposing the Hamiltonian constraint $H_{\text{reg}} = 0$, we obtain the following modified Friedmann equation

$$\frac{\dot{a}^2}{a^2} = \frac{2\rho}{\gamma^2}\left(1 - \frac{\rho}{\rho_c}\right), \tag{32}$$

where $\rho_c \equiv \frac{\gamma^2 c^2}{2\lambda^2\iota^2}$.

Finally, with this modified equation it is now easy to verify that the Big Rip singularity is avoided, see for instance [21]. What we have just seen in using the LQC paradigm is namely that, in order to avoid the Big Rip singularity, one only needs to make use of the classical regularized Hamiltonian (no quantization of the Hamiltonian is necessary, in principle, only one has to assume the discrete character of the geometry to construct the classical regularized Hamiltonian). This is somehow different from the effective formulation, where one must quantize the Hamiltonian in order to avoid the singularities. It is important to realize that, here, it is the regularization of the classical Hamiltonian which avoids the Big Rip singularity, rather than the quantum effects due to its quantization. This seems to have been overlooked in a number of papers, where it is claimed that quantum effects are in fact essential to avoid the singularity [17, 21, 22]. Note that, in those approximations, one already starts from the quantum version of the regularized Hamiltonian (a discrete theory) and then, using semi-classical techniques, coherent state expressions, etc, an effective Hamiltonian is obtained (a continuous theory)[18, 23–25] which, in fact, is in essence the Hamiltonian (30). This is perhaps the reason why it was plainly concluded there that quantum effects, provided by LQC, are actually the ones responsible for avoiding the singularity. We believe that this procedure can produce some confusion, because it seems that the only way to describe the features of LQC within a continuous theory is to consider the effective Hamiltonian obtained from the quantum version of the theory (see for example [26, 27]).

4. Conclusions

We have shown how our effective formulation of gravity, which interpolates in a way between the Wheeler–DeWitt approach and classical cosmology, is able to avoid both the classical Big Bang singularity and also the Big Rip singularity. Our formulation, is in essence, Schrödinger's equation with the condition that the average of the Hamiltonian operator be zero. Our formulation is different from the Wheeler–DeWitt equation where one imposes that the Hamiltonian operator annihilates the wavefunction, and where the arrow of time is yet to be selected. In our theory, we have assumed that time exists [3], and that it has the same meaning as in the classical theory, and the relevant quantities are averages of the quantum operators as, e.g. the average of the scale factor operator—which is by definition strictly positive—and there appears no Big Bang singularity at finite time. The Big Rip singularity problem is more involved and, in order to address and solve it one needs to work harder, as was seen in the last part of section 2 explicitly. It could turn out to be that the problem of time is associated with the avoidance of singularities. Our results hint toward this direction.

Another way to deal with the classical Big Rip singularity problem is loop quantum cosmology. We have shown, in this respect, that it is the regularization itself performed on the classical Hamiltonian what seems to avoid this singularity, rather than the quantum effects arising after the quantization of the regularized Hamiltonian. In this sense, one can actually say that the power to avoid the singularities lies in the principles themselves of the LQC paradigm.

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